

$$F \approx \tau \ln \left( - \left( - \frac{\hbar w}{\tau} \right) \right) = \tau \ln \frac{\hbar w}{\tau}$$

$$(8) G = - \left( \frac{\partial F}{\partial \tau} \right)_{V,N} = \left[ \ln \left( 1 - e^{-\hbar w/\tau} \right) + \tau \frac{-1}{1 - e^{-\hbar w/\tau}} e^{-\hbar w/\tau} \frac{\hbar w}{\tau^2} \right]$$

these don't change  
anyway!

$$= \left( \frac{\frac{\hbar w}{\tau}}{e^{\hbar w/\tau} - 1} - \ln \left( 1 - e^{-\hbar w/\tau} \right) \right) \text{ as required}$$

$$\left[ \text{Note that } \frac{e^{-\hbar w/\tau}}{1 - e^{-\hbar w/\tau}} = \frac{e^{-\hbar w/\tau} e^{\hbar w/\tau}}{e^{\hbar w/\tau} - e^{-\hbar w/\tau} e^{\hbar w/\tau}} = \frac{1}{e^{\hbar w/\tau} - 1} \right]$$

(4) Kittel 3.4

(12) in the book, Ch.3

$$\text{RHS of (89)} = \tau^2 \left( \frac{\partial U}{\partial \tau} \right)_{V,N} = \tau^2 \frac{\partial}{\partial \tau} \left( \tau^2 \frac{\partial \ln Z}{\partial \tau} \right)_{V,N} =$$

$$= \tau^2 \left[ 2\tau \frac{\partial \ln Z}{\partial \tau} + \tau^2 \frac{\partial^2 \ln Z}{\partial \tau^2} \right]_{V,N} = \tau^3 \left( 2 \frac{1}{Z} \frac{\partial Z}{\partial \tau} + \tau \frac{-1}{Z^2} \frac{\partial Z}{\partial \tau} \frac{\partial Z}{\partial \tau} + \tau \frac{1}{Z} \frac{\partial^2 Z}{\partial \tau^2} \right)$$

$$\text{LHS of (89)} = \langle (\varepsilon - \langle \varepsilon \rangle)^2 \rangle = \langle (\varepsilon - U)^2 \rangle = \langle \varepsilon^2 - 2U\varepsilon +$$

$$= \langle \varepsilon^2 \rangle - 2U \langle \varepsilon \rangle + U^2 = \langle \varepsilon^2 \rangle - U^2 = \frac{1}{Z} \sum_s \varepsilon_s^2 e^{-\varepsilon_s/\tau} - U^2$$

$$\text{Consider } \left( \frac{\partial Z}{\partial \tau} \right)_{V,N} = \frac{\partial}{\partial \tau} \sum_s e^{-\varepsilon_s/\tau} = \sum_s \frac{\varepsilon_s}{\tau^2} e^{-\varepsilon_s/\tau}$$

$$\text{Consider } \frac{\partial}{\partial \tau} \left( \tau^2 \frac{\partial Z}{\partial \tau} \right)_{V,N} = \frac{\partial}{\partial \tau} \sum_s \varepsilon_s e^{-\varepsilon_s/\tau} = \frac{1}{\tau^2} \sum_s \varepsilon_s^2 e^{-\varepsilon_s/\tau}$$

$$\text{Hence, LHS} \stackrel{(12)}{=} \frac{1}{Z} \tau^2 \frac{\partial}{\partial \tau} \left( \tau^2 \frac{\partial Z}{\partial \tau} \right) - \left( \tau^2 \frac{\partial \ln Z}{\partial \tau} \right)^2 =$$

$$= \frac{1}{Z} \tau^2 \left( 2\tau \frac{\partial Z}{\partial \tau} + \tau^2 \frac{\partial^2 Z}{\partial \tau^2} \right) - \tau^4 \cancel{\left( \frac{1}{Z} \frac{\partial Z}{\partial \tau} \right)^2} =$$

$$= \frac{1}{Z} \tau^3 \left( 2 \frac{\partial Z}{\partial \tau} + \tau \frac{\partial^2 Z}{\partial \tau^2} - \tau^2 \frac{1}{Z} \left( \frac{\partial Z}{\partial \tau} \right)^2 \right) = \text{RHS and we're done.}$$

It's actually easier to differentiate with respect to  $\beta = \frac{1}{\tau}$ .